

ADDITIVE SET FUNCTIONS ON LATTICES OF SETS⁽¹⁾

BY

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ABSTRACT. This paper is concerned with properties of additive set functions defined on lattices of sets. Extensions of results of Brunk and Johansen, Darst, Johansen, and Uhl are obtained. Two fundamental approximation properties for lattices of sets (established in another paper) permit us to translate the setting and consider countably additive set functions defined on sigma lattices of sets. Thereby results for countably additive set functions defined on sigma lattices of sets are used to obtain alternate derivations and extensions of Darst's results for additive set functions defined on lattices of sets, i.e., we consider the Radon-Nikodym derivative, conditional expectation, and martingale convergence for lattices of sets.

1. Introduction. In this paper we consider additive set functions defined on lattices of sets. We shall extend results of Brunk and Johansen, Darst, Johansen, and Uhl, obtained in [2], [3], [8], and [14]. The two approximation theorems for lattices of sets obtained in [5] permit us to translate the setting (§ III) and consider σ -additive set functions defined on σ -lattices of sets. Thereby we are able to use the results obtained in [4] and results of Johansen for σ -additive set functions defined on σ -lattices of sets to obtain alternate derivations and extensions of Darst's results for additive set functions defined on lattices of sets, i.e., we discuss the Radon-Nikodym derivative (§ IV), conditional expectation (§ V), and martingale convergence (§ VI) in this setting. We state our results in the setting of the Orlicz spaces $L^\Phi(\Omega, \mathcal{U}, \mu, R)$ and $V^\Phi(\Omega, \mathcal{U}, \mu, R)$ as defined by Uhl in [14], where \mathcal{U} is an algebra of subsets of Ω and we assume that μ is a nonnegative bounded additive function on \mathcal{U} . In § VII we extend our results to include vector valued (R^n) set functions.

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II. Preliminaries. We begin by recalling some basic definitions.

Definition 2.1. A class \mathfrak{M} of subsets of a nonempty set Ω is called a lattice of subsets of Ω if both the empty set \emptyset and Ω belong to \mathfrak{M} and both $A \cup B$ and $A \cap B$ are in \mathfrak{M} whenever A and B are elements of \mathfrak{M} . If in addition \mathfrak{M} is closed under countably infinite unions and intersections, then \mathfrak{M} is called a σ -lattice.

For example, in a metric space both the class of open sets and the class of closed sets are lattices of sets.

Definition 2.2. A real valued set function μ on \mathfrak{M} is said to be additive if

$$(2.2.1) \quad \mu(\emptyset) = 0 \text{ and}$$

$$(2.2.2) \quad \mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B), \quad A, B \in \mathfrak{M}.$$

The function μ is said to be σ -additive (countably additive) if, in addition, for $A_i \cap A_j = \emptyset$, $i \neq j$,

$$(2.2.3) \quad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n), \quad A_n \in \mathfrak{M}, \quad n = 1, 2, \dots$$

Throughout this paper, \mathfrak{M} will denote a lattice of subsets of a nonempty set Ω .

Suppose each of λ and μ is a bounded, additive function on \mathfrak{M} and μ is non-negative (i.e. $\mu(A) \leq \mu(B)$ if $A \subset B$).

Results of B. J. Pettis [11, Corollary 1.2.1, p. 189] assert that a real valued function F defined on a lattice \mathfrak{M} has a unique additive extension to the algebra \mathfrak{A} generated by \mathfrak{M} if, and only if, F satisfies (2.2.1) and (2.2.2). Thus, each of λ and μ has a unique additive extension to \mathfrak{A} . These extensions will also be denoted by λ and μ . Moreover, the conditions (2.2.1) and

$$(2.2.4) \quad \mu(A \cup B) = \mu(A) + \mu(B), \quad A, B \in \mathfrak{M}, \quad A \cap B = \emptyset,$$

do not imply (2.2.2).

Definition 2.3. An extended real valued function f on Ω is said to be \mathfrak{M} -measurable if $[f > a] \in \mathfrak{M}$, $a \in R$; if the range of f is a finite set then f is called a simple function.

For example, the bounded upper semicontinuous functions on $I = [0, 1]$ are uniform limits of simple upper semicontinuous functions which are measurable with respect to the lattice of closed subsets of I .

Suppose that \mathfrak{M} is a sublattice of an algebra \mathfrak{U} of subsets of Ω and that μ is a nonnegative bounded additive function on \mathfrak{U} .

In order to state the results in the following sections succinctly, we recall some basic facts about Orlicz spaces. Suppose that Φ is a continuous Young's function [14, p. 20] satisfying the Δ_2 -condition (Φ satisfies the Δ_2 -condition if there exists $0 < K < \infty$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \in R$). Then Φ is an

even, convex function defined on the real numbers R with $\Phi(0) = 0$ and $\Phi(x) \neq 0$ for some x . Examples are $\Phi(x) = |x|^p$, $1 \leq p < \infty$.

The Orlicz space $L^\Phi(\Omega, \mathcal{U}, \mu, B)$, B a Banach space, is the space of equivalence classes of totally μ -measurable functions f on Ω such that

$$\int_{\Omega} \Phi(\|f(x)\|) d\mu < \infty,$$

where totally μ -measurable and integration are as in Chapter III of Dunford and Schwartz [6]. By [9, Theorem 9.4, p. 76], if Φ satisfies the Δ_2 -condition, and $B = R$, then convergence in norm means

$$\lim_{n \rightarrow \infty} \int_{\Omega} \phi(\|f(x) - f_n(x)\|) d\mu = 0.$$

The completion $V^\Phi(\Omega, \mathcal{U}, \mu, B)$ of $L^\Phi(\Omega, \mathcal{U}, \mu, B)$ is a space of μ -continuous additive set functions on \mathcal{U} . Information about these L^Φ and V^Φ spaces can be found in [14] and the references therein. By [14, Corollary 7, p. 49], step functions (indefinite integrals of simple functions) are dense in V^Φ . Hence $H \in V^\Phi$ can be identified with a sequence $\{b_n\}_{n=1}^\infty$ of simple \mathcal{U} -measurable functions and

$$\|H\|_\Phi = \lim_{n \rightarrow \infty} \|b_n\|_\Phi.$$

If $\Phi(x) = |x|^p$, $p \geq 1$, then we denote L^Φ by L_p and V^Φ by V_p . Since μ is finite, $L^\Phi \subset L_1$, $V^\Phi \subset V_1$, and

$$\|\cdot\|_1 \leq M \|\cdot\|_\Phi.$$

III. Applying the isomorphisms τ , T , and U to translate the setting. Suppose each of λ and μ is a (real valued) bounded, additive function on \mathfrak{M} and μ is nonnegative, and that \mathcal{Q} is the algebra generated by \mathfrak{M} . Recall that we are also denoting the unique extensions of λ and μ to \mathcal{Q} by λ and μ .

Referring the reader to pp. 97, 164–167, 240, 274–276, and 311–313 of Dunford and Schwartz [6] for details, we recall that there is a totally disconnected compact Hausdorff space Ω_1 and an isomorphism τ of \mathcal{Q} onto the algebra \mathcal{Q}_1 of closed and open subsets of Ω_1 such that the images λ_1 and μ_1 of λ and μ (i.e. $\mu_1(\tau(A)) = \mu(A)$) are countably additive on \mathcal{Q}_1 . Consequently, λ_1 and μ_1 have unique countably additive extensions λ_2 and μ_2 to the σ -algebra \mathcal{Q}_2 generated by \mathcal{Q}_1 . The mappings $T: \mu \rightarrow \mu_1$ and $U: \mu_1 \rightarrow \mu_2$ are isometric isomorphisms.

Let $\mathfrak{M}_1 = \tau(\mathfrak{M}) = \{A_1: A_1 = \tau(A), A \in \mathfrak{M}\}$. Then \mathfrak{M}_1 is a lattice of subsets of Ω_1 and the algebra \mathcal{Q}_1 is the algebra generated by \mathfrak{M}_1 . The lattice \mathfrak{M}_1 generates a σ -lattice \mathfrak{M}_2 which is contained in the σ -algebra \mathcal{Q}_2 . But λ_2 and μ_2 are countably additive on \mathcal{Q}_2 , and hence on \mathfrak{M}_2 .

The approximation theorems for lattices of sets established in [5] (\mathfrak{M}_1 is dense in \mathfrak{M}_2 and \mathfrak{M}_2 -measurable functions can be approximated by simple \mathfrak{M}_1 -

measurable functions in $L^\Phi(\Omega_1, \mathfrak{Q}_2, \mu_2, R)$ and the isomorphisms τ , T , and U are used in the following sections to obtain results for λ and μ on \mathfrak{M} from the corresponding results for λ_2 and μ_2 on \mathfrak{M}_2 . Uhl [14] used these isomorphisms to obtain the derivative of λ with respect to μ on \mathfrak{Q} .

IV. The Radon-Nikodym derivative given a lattice of sets. We are considering real valued bounded additive set functions, λ and μ , defined on a lattice \mathfrak{M} of sets. The following example illustrates the fact that even if λ and μ are countably additive, \mathfrak{M} is a σ -lattice, and λ is absolutely continuous with respect to μ , it may not be possible to approximate λ by integrals of simple \mathfrak{M} -measurable functions. Recall that f is \mathfrak{M} -measurable if $[f > a] \in \mathfrak{M}$ for all $a \in R$.

Example 4.1. Suppose $\Omega = \{a, b\}$, $\mathfrak{M} = \{\emptyset, \{a\}, \Omega\}$, $\mathfrak{Q} = \{\emptyset, \{a\}, \{b\}, \Omega\}$, $\lambda(\emptyset) = 0$, $\lambda(\{a\}) = 2$, $\lambda(\{b\}) = 4$, $\lambda(\Omega) = 6$, $\mu(\emptyset) = 0$, $\mu(\{a\}) = \mu(\{b\}) = 1$, and $\mu(\Omega) = 2$. Then for f to be \mathfrak{M} -measurable, $f(a) \geq f(b)$, because if $f(a) < c < f(b)$, then $[f > c] = \{b\} \notin \mathfrak{M}$. Therefore, if $F(E) = \int_E f d\mu$, where f is a simple \mathfrak{M} -measurable function, then $F(\{a\}) \geq F(\{b\})$. But $\lambda(\{a\}) = \lambda(\{b\}) - 2$, thus it is impossible to approximate λ by integrals of simple \mathfrak{M} -measurable functions. Moreover, suppose $f(a) = 2$ and $f(b) = -3$; then $f = 2\chi_{\{a\}} + (-3)\chi_{\{b\}}$. Hence, f is a simple \mathfrak{M} -measurable function, but we cannot write $f = \sum x_n \chi_{E_n}$, where $E_n \in \mathfrak{M}$ and E_n is the set on which f takes on the value x_n . Notice also that the functions $-f$, $|f|$, and f^2 are not \mathfrak{M} -measurable.

Nevertheless, Radon-Nikodym derivatives have been defined for additive set functions on lattices of sets.

In [8], S. Johansen defined, constructed, and established the uniqueness of a Radon-Nikodym derivative for σ -additive set functions defined on σ -lattices of sets. We shall obtain information about λ and μ by applying Johansen's results to λ_2 and μ_2 (where λ , μ , λ_2 , and μ_2 are as in § III).

In [3], R. B. Darst established the analogous definition, construction, and uniqueness of a Radon-Nikodym derivative for nonnegative, bounded, finitely additive set functions defined on lattices of sets. For the sake of completeness, we state the definition for the case where λ is not assumed to be nonnegative. The appearance of M in the definition is a technical inconvenience made necessary by the fact that $[f_n \leq b] = \Omega$ and $[f_n > a] = \Omega$ for fixed n and large positive b and large negative a .

Definition 4.2. We call a sequence $\{f_n\}$ of simple \mathfrak{M} -measurable functions defined on Ω a derivative (or a "sequence of approximate derivatives") of λ with respect to μ on \mathfrak{M} if $\{f_n\}$ is a L_1 -Cauchy sequence: $\int_\Omega |f_m - f_n| d\mu \rightarrow 0$, and, for every $\epsilon > 0$, $M > 0$, there is a positive integer N such that if $n > N$, $a, b \in R$, $b < M$, $a > -M$, and $A, B \in \mathfrak{M}$, then

$$(4.2.1) \quad \lambda(A \cap [f_n \leq b]) < b\mu(A \cap [f_n \leq b]) + \epsilon,$$

$$(4.2.2) \quad \lambda(B^C \cap [f_n > a]) > a\mu(B^C \cap [f_n > a]) - \epsilon.$$

In Example 4.1, the function f defined by $f(a) = f(b) = 3$ is the Radon-Nikodym derivative of λ with respect to μ on \mathfrak{M} . However, the function g defined by $g(a) = 2$ and $g(b) = 4$ is the classical Radon-Nikodym derivative of λ with respect to μ , and $f \neq g$ everywhere. The function g is not \mathfrak{M} -measurable.

We now turn to the construction of the Radon-Nikodym derivative of λ with respect to μ on \mathfrak{M} . We assume that both λ and μ are nonnegative.

Suppose we let $\Phi(x) = |x|$, $x \in R$. Then $L = L_1$, and Johansen's derivative $f \in L_1$ of λ_2 with respect to μ_2 on \mathfrak{M}_2 can be approximated [5, Theorem 2] by a L_1 -Cauchy sequence $\{f_n''\}$ of simple \mathfrak{M}_1 -measurable functions. In the proof of [5, Theorem 2], the interval $[0, n]$ is split into $n2^n$ equal parts of lengths $1/2^n$ and then the simple \mathfrak{M}_1 -measurable function f_n'' is defined by

$$f_n'' = \begin{cases} n & \text{on } E_0, \\ n - j/2^n & \text{on } E_j \cap \left(\bigcup_{i=0}^{j-1} E_i \right)^c, \quad j = 1, \dots, n2^n, \end{cases}$$

where $E_j \in \mathfrak{M}_1$ (applying [5, Theorem 1]) is close to $[f > n - j/2^n]$, $j = 0, 1, \dots, n2^n - 1$, and E_{n2^n} is close to $[f \geq 0]$, in measure (we can use $\lambda_2 + \mu_2$ instead of μ_2). If we let $\tau(A_j) = E_j$, then by the inverse image of f_n'' , under τ , we mean the simple \mathfrak{M} -measurable function f_n defined by

$$f_n = \begin{cases} n & \text{on } A_0, \\ n - j/2^n & \text{on } A_j \cap \left(\bigcup_{i=0}^{j-1} A_i \right)^c, \quad j = 1, \dots, n2^n. \end{cases}$$

Since $\mu(A_j) = \mu_1(\tau(A_j)) = \mu_1(E_j) = \mu_2(E_j)$, the functions $|f_n - f_m|$ and $|f_n'' - f_m''|$ are finitely valued and have the same values on sets of the same measure. Consequently,

$$\int_{\Omega} |f_n - f_m| d\mu = \int_{\Omega_1} |f_n'' - f_m''| d\mu_2.$$

Therefore $\{f_n\}$ is a L_1 -Cauchy sequence of simple \mathfrak{M} -measurable functions, and is, in fact, the Radon-Nikodym derivative of λ with respect to μ on \mathfrak{M} (i.e., $\{f_n\}$ satisfies the requirements of Definition 4.2).

Theorem 4.3. *The sequence $\{f_n\}$ is the derivative of λ with respect to μ on \mathfrak{M} .*

Proof. First show that the sequence $\{f_n''\}$ satisfies the requirements of Definition 4.2 for λ_2 and μ_2 on \mathfrak{M}_2 . We omit computations. Then to verify that the sequence $\{f_n\}$ satisfies (4.2.1), simply note that

$$\begin{aligned} \lambda(A \cap [f_n \leq b]) &= \lambda_2(\tau A \cap [f_n'' \leq b]) \\ &\leq b\mu_2(\tau A \cap [f_n'' \leq b]) + \epsilon = b\mu(A \cap [f_n \leq b]) + \epsilon. \end{aligned}$$

Similarly, the sequence $\{f_n\}$ satisfies (4.2.2). Thus $\{f_n\}$ is the Radon-Nikodym derivative of λ with respect to μ on \mathfrak{M} .

V. Conditional expectation given a lattice of sets. Recall that if $b \in L_1(\Omega, \mathfrak{A}, \mu)$, where $(\Omega, \mathfrak{A}, \mu)$ is a probability space, and \mathfrak{B} is a sub- σ -algebra of the σ -algebra \mathfrak{A} , then $f = E(b, \mathfrak{B})$ is the conditional expectation of b given \mathfrak{B} if f is \mathfrak{B} -measurable and $\int_A f d\mu = \int_A b d\mu$ for all $A \in \mathfrak{B}$. If $b \in L_2(\Omega, \mathfrak{A}, \mu)$, then $f \in L_2(\Omega, \mathfrak{A}, \mu)$ and f minimizes the expression $\int_{\Omega} (b-k)^2 d\mu$, where k is \mathfrak{B} -measurable and $k \in L_2(\Omega, \mathfrak{A}, \mu)$, i.e., f is the best \mathfrak{B} -measurable L_2 -approximation to b .

H. D. Brunk defined conditional expectation given a σ -lattice, \mathfrak{M}_2 , and established several of its properties in [1]. Denote the conditional expectation of $b \in L_1$ given \mathfrak{M}_2 by $E(b, \mathfrak{M}_2)$. For $b \in L_2$, Brunk characterized $E(b, \mathfrak{M}_2)$ as the unique \mathfrak{M}_2 -measurable best L_2 -approximation to b , and discussed applications in which $E(b, \mathfrak{M}_2)$ is the solution to maximum likelihood estimation problems. Subsequently S. Johansen [8] described a Radon-Nikodym derivative given a σ -lattice and showed (Theorem 5.6) that the Radon-Nikodym derivative was the conditional expectation described by Brunk in the appropriate case (L_2). Similarly, R. B. Darst [3] showed (Theorem 5.5) that his Radon-Nikodym derivative given a lattice \mathfrak{M} (Definition 4.2) was the best \mathfrak{M} -measurable approximation in the appropriate case (V_2). Using the isomorphisms τ , T , and U , and the second approximation theorem for lattices, [5, Theorem 2], and applying Johansen's result (Theorem 5.6) for σ -lattices, we are able to obtain a much more transparent proof of Darst's result (Theorem 5.5) for lattices. We first show that the derivative of a V^Φ -function is a V^Φ -function.

Definition 5.1. If λ is the restriction to \mathfrak{M} of an element $H = \{b_n\}_{n=1}^\infty$ of $V^\Phi(\Omega, \mathfrak{U}, \mu, R)$, then we shall denote the Radon-Nikodym derivative (Definition 4.2), $F = \{f_n\}_{n=1}^\infty$, of λ with respect to μ on \mathfrak{M} by $F = E(H, \mathfrak{M})$.

Definition 5.2. If $b \in L^\Phi(\Omega_1, \mathfrak{U}_2, \mu_2, R)$ and $H_2(E) = \int_E b d\mu_2$, $E \in \mathfrak{U}_2$, then we shall denote the derivative [8] f of H_2 with respect to μ_2 on \mathfrak{M}_2 by $f = E(b, \mathfrak{M}_2)$.

The following theorem for σ -lattices is established in [4].

Theorem 5.3. *The Radon-Nikodym derivative of an L^Φ -function b is an L^Φ -function $f = E(b, \mathfrak{M}_2)$.*

Theorem 5.4. *The Radon-Nikodym derivative of a V^Φ -function H is a V^Φ -function $F = E(H, \mathfrak{M})$.*

Proof. Suppose $H = \{b_n\}$ is in $V^\Phi(\Omega, \mathfrak{U}, \mu, R)$. By the isomorphisms λ , T , and U ,

$$\begin{aligned} \mathfrak{U} &\rightarrow \mathfrak{U}_1 \rightarrow \mathfrak{U}_2, & \mathfrak{A} &\rightarrow \mathfrak{A}_1 \rightarrow \mathfrak{A}_2, & \mathfrak{M} &\rightarrow \mathfrak{M}_1 \rightarrow \mathfrak{M}_2, \\ H &\rightarrow H_1 \rightarrow H_2, & \mu &\rightarrow \mu_1 \rightarrow \mu_2. \end{aligned}$$

If $b_n = \sum \alpha_k \chi_{E_k}$, let $b_n'' = b_n' = \sum \alpha_k \chi_{\tau E_k}$. Then, since $H(E) = \lim_{n \rightarrow \infty} \int_E b_n d\mu$ uniformly for all $E \in \mathcal{U}$, and $H_1(\tau E) = H(E)$, it follows that

$$H_1(E) = \lim_{n \rightarrow \infty} \int_E b_n' d\mu_1 \quad \text{uniformly for all } E \in \mathcal{U}_1.$$

Setting $\gamma_n(E) = \int_E b_n d\mu$ on \mathcal{U} , $\gamma_n'(E) = \int_E b_n' d\mu_1$ on \mathcal{U}_1 , and $\gamma_n''(E) = \int_E b_n'' d\mu_2$ on \mathcal{U}_2 , we get

$$\|H_2 - \gamma_n''\|_1 = \|H_1 - \gamma_n'\|_1 = \|H - \gamma_n\|_1 \rightarrow 0$$

because U and T are isometric isomorphisms. Thus,

$$(5.4.1) \quad H_2(E) = \lim_{n \rightarrow \infty} \int_E b_n'' d\mu_2 \quad \text{uniformly for all } E \in \mathcal{U}_2.$$

Since $\int_{\Omega_1} \Phi(b_n'' - b_m'') d\mu_2 = \int_{\Omega} \Phi(b_n - b_m) d\mu \rightarrow 0$, and the space $L^\Phi(\Omega_1, \mathcal{U}_2, \mu_2, R)$ is complete, there is a unique $b \in L^\Phi(\Omega_1, \mathcal{U}_2, \mu_2, R)$ such that b_n'' converges to b in L^Φ -norm. But $\|b_n'' - b\|_1 \leq M \|b_n'' - b_n'\|_\Phi$; therefore

$$(5.4.2) \quad \int_E b d\mu_2 = \lim_{n \rightarrow \infty} \int_E b_n'' d\mu_2 \quad \text{uniformly for all } E \in \mathcal{U}_2.$$

It follows from (5.4.1) and (5.4.2) that

$$(5.4.3) \quad H_2(E) = \int_E b d\mu_2 \quad \text{for all } E \in \mathcal{U}_2.$$

By Theorem 5.3, $f = E(b, \mathfrak{M}_2)$ is in $L^\Phi(\Omega_1, \mathcal{U}_2, \mu_2, R)$. Thus, by [5, Theorem 2], there exists a sequence $\{f_n''\}$ of simple \mathfrak{M}_1 -measurable functions such that $\lim_{n \rightarrow \infty} \int_{\Omega_1} \Phi(f - f_n'') d\mu_2 = 0$ (which implies $\lim_{n \rightarrow \infty} \int_{\Omega_1} |f - f_n''| d\mu_2 = 0$). Then the inverse image under τ of the sequence $\{f_n''\}$ is a sequence $F = \{f_n\}$ of simple \mathfrak{M} -measurable functions, and by Theorem 4.3, $F = E(H, \mathfrak{M})$. Finally,

$$\int_{\Omega} \Phi(f_n - f_m) d\mu = \int_{\Omega_1} \Phi(f_n'' - f_m'') d\mu_2 \rightarrow 0,$$

hence $F \in V^\Phi(\Omega, \mathcal{U}, \mu, R)$.

For $H \in V_2(\Omega, \mathcal{U}, \mu, R)$, the following theorem characterizes $F = E(H, \mathfrak{M})$ as the best V_2 -approximation to H by a L_2 -Cauchy sequence of simple \mathfrak{M} -measurable functions.

Theorem 5.5 (Darst). *If $H \in V_2(\Omega, \mathcal{U}, \mu, R)$ and $F = E(H, \mathfrak{M})$, then*

$$(5.5.1) \quad F \in V_2,$$

$$(5.5.2) \quad \int HK d\mu \leq \int FK d\mu, \quad K = \{k_m\} \text{ } \mathfrak{M}\text{-measurable, } K \in V_2,$$

$$(5.5.3) \quad \int HK d\mu \geq \int FK d\mu, \quad K = \{k_m\} \text{ } \mathfrak{M}^c\text{-measurable, } K \in V_2$$

$$(\mathfrak{M}^c = \{A: A^c \in \mathfrak{M}\}),$$

$$(5.5.4) \quad \int HF d\mu = \int F^2 d\mu, \quad \text{and}$$

$$(5.5.5) \quad \int (H - F)^2 d\mu \leq \int (H - K)^2 d\mu, \quad K = \{k_m\} \text{ } \mathfrak{M}\text{-measurable, } K \in V_2.$$

Moreover, the conditions of the theorem characterize F among the L_2 -Cauchy sequences.

Before giving our proof of Theorem 5.5, we give a geometric interpretation. Brunk [1] used this interpretation to define $f = E(b, \mathfrak{M}_2)$, $b \in L_2$.

Let $V_2(\mathfrak{M})$ denote the class of \mathfrak{M} -measurable $K = \{k_m\}$ in $V_2(\Omega, \mathfrak{U}, \mu, R)$. If $A(V_2(\mathfrak{M}))$ is a closed convex cone in a complete inner product space $B(V_2(\Omega, \mathfrak{U}, \mu, R))$ with $(H, F) = \int HF d\mu = \lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} \int b_m f_n d\mu)$, and $H \in B$, then there is a unique closest point F of A to H , hence (5.5.1) and (5.5.5) and the last assertion of the theorem. Brunk [1, Corollary 2.3, p. 1340] used the notation $F = P(H|A)$ (P for projection) and proved that $F = P(H|A)$ if, and only if,

(i) $(H - F, F) = 0$ ($\int HF d\mu = \int F^2 d\mu$), i.e., the vector $H - F$ is orthogonal to the vector F , and

(ii) $(H - F, K) \leq 0$ for all $K \in A$ ((5.5.2), and also (5.5.3) because if K is \mathfrak{M}^c -measurable, then $-K$ is \mathfrak{M} -measurable), i.e., the angle between the vector $H - F$ and the vector K is greater than or equal to 90° .

Turning now to the proof of Theorem 5.5, consider the proof of Theorem 5.4 with $\Phi(x) = |x|^2$. Then by (5.4.3), $H_2(E) = \int_E b d\mu_2$ for all $E \in \mathfrak{U}_2$, where $b \in L_2(\Omega_1, \mathfrak{U}_2, \mu_2, R)$. Consequently, we can apply Johansen's result [8, Theorem 5, p. 55] for σ -lattices:

Theorem 5.6 (Johansen). *If $b \in L_1 \cap L_2(\Omega_1, \mathfrak{U}_2, \mu_2, R)$, then $f = E(b, \mathfrak{M}_2)$ is an \mathfrak{M}_2 -measurable function on Ω_1 which is characterized by the following conditions:*

$$(5.6.1) \quad f \in L_1 \cap L_2.$$

$$(5.6.2) \quad \int bk d\mu_2 \leq \int fk d\mu_2, \quad k \text{ } \mathfrak{M}_2\text{-measurable, } k \in L_1 \cap L_2,$$

$$(5.6.3) \quad \int bk d\mu_2 \geq \int fk d\mu_2, \quad k \text{ } \mathfrak{M}_2^c\text{-measurable, } k \in L_1 \cap L_2.$$

$$(5.6.4) \quad \int bk d\mu_2 = \int f^2 d\mu_2.$$

Furthermore,

$$(5.6.5) \quad \int (b - f)^2 d\mu_2 \leq \int (b - k)^2 d\mu_2, \quad k \text{ } \mathfrak{M}_2\text{-measurable, } k \in L_1 \cap L_2.$$

With $\Phi(x) = |x|^2$, the proof of Theorem 5.4 yields

$$\int_{\Omega_1} |b_n'' - b|^2 d\mu_2 \rightarrow 0, \quad \int_{\Omega_1} |f_n'' - f|^2 d\mu_2 \rightarrow 0,$$

and

$$\int_{\Omega} |f_n - f_m|^2 d\mu = \int_{\Omega_1} |f_n'' - f_m''|^2 d\mu_2 \rightarrow 0.$$

Thus, $F = \{f_n\} = E(H, \mathfrak{M})$ is in $V_2(\Omega, \mathfrak{U}, \mu, R)$.

We obtain (5.5.2) and (5.5.3) by applying (5.6.2) and (5.6.3) to

$$\int HK d\mu = \lim_n \left(\lim_m \int b_m k_n d\mu \right) = \lim_n \left(\lim_m \int b_m'' k_n'' d\mu_2 \right) = \lim_n \left(\int b k_n'' d\mu_2 \right)$$

and

$$\int FK d\mu = \lim_n \left(\lim_m \int f_m k_n d\mu \right) = \lim_n \left(\lim_m \int f_m'' k_n'' d\mu_2 \right) = \lim_m \left(\int f k_n'' d\mu_2 \right)$$

respectively.

Since $\int HF d\mu = \int b f d\mu_2$ and $\int F^2 d\mu = \int f^2 d\mu_2$, (5.6.4) implies (5.5.4).

The last property, (5.5.5), now follows from

$$(A) \quad \int (H - K)^2 d\mu = \int (H - F)^2 d\mu + \int (F - K)^2 d\mu + 2 \int (H - F)(F - K) d\mu \geq 0$$

and

$$\int (H - F)(F - K) d\mu = \int HF d\mu - \int F^2 d\mu - \int HK d\mu + \int FK d\mu \geq 0.$$

Moreover, (A) also shows that if K can play the role of F in the theorem, then $\int (F - K)^2 d\mu = 0$. Hence, the conditions of the theorem characterize the Radon-Nikodym derivative of H on \mathfrak{M} .

VI. A martingale convergence theorem for lattices. It is, perhaps, appropriate to begin our discussion of martingale convergence with references to some recent work.

In [12], J.J. Uhl discussed convergence in the Orlicz space $L^\Phi(\Omega, \mathfrak{A}, \mu, B)$ (\mathfrak{A} a σ -algebra and μ σ -additive) of Banach space valued martingales $\{f_t, \mathfrak{A}_t\}$, where $\{\mathfrak{A}_t\}$ is an increasing net of sub- σ -algebras and $f_{t_1} = E(f_{t_2}, \mathfrak{A}_{t_1})$ for $t_2 \geq t_1$. In [13], Uhl discussed convergence in the Orlicz space $V^\Phi(\Omega, \mathfrak{A}, \mu, B)$ (\mathfrak{A} an algebra and μ finitely additive) of martingales $\{F_t, \mathfrak{A}_t\}$ of Banach space valued additive set functions F_t , where $\{\mathfrak{A}_t\}$ is an increasing net of subalgebras and $F_{t_1} = E(F_{t_2}, \mathfrak{A}_{t_1})$ for $t_2 \geq t_1$. Uhl [13] noted that some of the results he obtained for additive set functions defined on algebras of sets can be deduced from known results for σ -additive set functions defined on σ -algebras of sets by using the isomorphisms τ , T , and U .

H. D. Brunk and S. Johansen [2, Theorem 2.8, p. 605] proved an almost everywhere martingale convergence theorem for the Radon-Nikodym derivatives given an increasing sequence of σ -lattices. Norm convergence (in the Orlicz space

$L^\Phi(\Omega, \mathcal{Q}, \mu, R)$, \mathcal{Q} a σ -algebra and μ σ -additive) of these derivatives was established in [4, Theorem 2].

In [3], R. B. Darst established which features of a martingale convergence theorem carry over to the setting of additive set functions defined on lattices of sets: proving convergence in $V_1(\Omega, \mathcal{Q}, \mu, R)$ of martingales of Radon-Nikodym derivatives given a lattice (Definition 4.4). Using the isomorphisms τ , T , and U and applying the corresponding martingale convergence theorem for σ -lattices, [4, Theorem 2], we are able to obtain an extension of Darst's result to $V^\Phi(\Omega, \mathcal{U}, \mu, R)$. We begin by establishing the necessary notation.

Suppose that $\{\mathcal{M}(k)\}$ is an increasing sequence of lattices of subsets of Ω , and $\mathcal{M} = \bigcup_k \mathcal{M}(k)$. Then the algebras $\mathcal{Q}(k)$ of subsets of Ω that are generated by these lattices increase to the algebra \mathcal{Q} generated by the lattice \mathcal{M} .

Suppose that λ and μ are nonnegative bounded additive functions on \mathcal{Q} , and that $\lambda \in V^\Phi(\Omega, \mathcal{Q}, \mu, R)$. Denote by $\lambda(k)$ and $\mu(k)$ the restrictions of λ and μ to $\mathcal{Q}(k)$. Then $\lambda(k)$ is in $V^\Phi(\Omega, \mathcal{Q}(k), \mu(k), R)$. Let $E(\lambda, \mathcal{M}) = F = \{f_n\}_{n=1}^\infty$ and $E(\lambda(k), \mathcal{M}(k)) = G_k = \{g_{k,n}\}_{n=1}^\infty$. By Theorem 5.4, F and G_k are in V^Φ .

Because the sequences $\{g_{k,n}\}_{n=1}^\infty$ are Cauchy sequences in $L^\Phi(\Omega, \mathcal{Q}(k), \mu(k), R)$, they are Cauchy sequences in $L^\Phi(\Omega, \mathcal{Q}, \mu, R)$ (i.e. $\int_\Omega \Phi(g_{k,n} - g_{k,m}) d\mu = \int_\Omega \Phi(g_{k,n} - g_{k,m}) d\mu(k)$). Hence the equation

$$H_k(E) = \lim_{n \rightarrow \infty} \int_E g_{k,n} d\mu, \quad \text{for } E \in \mathcal{Q},$$

defines an additive extension of G_k to \mathcal{Q} . Notice that $\{H_k\}$ is determined by $\{\mathcal{M}(k)\}$, λ , and μ . We call $\{H_k, \mathcal{M}(k)\}$ a martingale, and prove the following martingale convergence theorem.

Theorem 6.1. *The sequence $\{H_k\}$ converges to F in $V^\Phi(\Omega, \mathcal{Q}, \mu, R)$.*

Remark. By assuming $\lambda \in V^\Phi$, we are assuming λ is absolutely continuous with respect to μ . Note that λ is absolutely continuous with respect to μ if, and only if, the $\lambda(k)$'s are uniformly absolutely continuous with respect to the $\mu(k)$'s (i.e. for each $\epsilon > 0$ there is a $\delta > 0$ such that if $E \in \mathcal{Q}(k)$ and $\mu(k)(E) < \delta$, then $\lambda(k)(E) < \epsilon$). An example illustrating this fact and which also shows that the requirement that λ be absolutely continuous with respect to μ is not superfluous can be found in [3, p. 597].

Proceeding to the proof of the theorem, apply the isomorphisms τ , T , and U to obtain

$$\begin{aligned} \mathcal{Q} &\rightarrow \mathcal{Q}_1 \rightarrow \mathcal{Q}_2, & \mathcal{M}(k) &\rightarrow \mathcal{M}(k, 1) \rightarrow \mathcal{M}(k, 2), \\ \mathcal{M} &\rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_2, & \lambda &\rightarrow \lambda_1 \rightarrow \lambda_2, \\ \mathcal{Q}(k) &\rightarrow \mathcal{Q}(k, 1) \rightarrow \mathcal{Q}(k, 2), & \mu &\rightarrow \mu_1 \rightarrow \mu_2. \end{aligned}$$

Then define $\lambda(k, i) = \lambda_i|_{\mathcal{Q}(k, i)}$ and $\mu(k, i) = \mu_i|_{\mathcal{Q}(k, i)}$ for $i = 1, 2$.

Appealing to the proof of Theorem 5.4, in particular (5.4.3), we may assume that

$$\lambda_2(E) = \int_E h d\mu_2, \quad h \in L^\Phi(\Omega_1, \mathcal{Q}_2, \mu_2, R),$$

and

$$\lambda(k, 2)(E) = \int_E b_k d\mu_2, \quad b_k \in L^\Phi(\Omega_1, \mathcal{Q}(k, 2), \mu(k, 2), R).$$

Setting $f = E(b, \mathbb{M}_2)$ and $d_k = E(b_k, \mathbb{M}(k, 2))$, it follows from the corresponding martingale convergence theorem for σ -lattices, [4, Theorem 2], that

$$\lim_{k \rightarrow \infty} \int_{\Omega_1} \Phi(f - d_k) d\mu_2 = 0.$$

Recall (see discussion preceding Theorem 4.3) that the sequences $\{g_{k,n}\} = H_k$ and $\{f_n\} = F$ are the inverse images under τ of sequences $\{g''_{k,n}\}$ and $\{f''_n\}$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega_1} \Phi(d_k - g''_{k,n}) d\mu_2 = 0, \text{ and}$$

$$\lim_{n \rightarrow \infty} \int_{\Omega_1} \Phi(f - f''_n) d\mu_2 = 0.$$

Thus,

$$\|H_k - F\|_\Phi = \lim_{n \rightarrow \infty} \|g_{k,n} - f_n\|_\Phi = \lim_{n \rightarrow \infty} \|g''_{k,n} - f''_n\|_\Phi = \|d_k - f\|_\Phi,$$

which completes the proof of Theorem 6.1.

Recalling [14] that $\lambda \in V^\Phi(\Omega, \mathcal{Q}, \mu, R)$ if, and only if, $I_\Phi(\lambda) < \infty$ and λ is μ -continuous, and that λ is μ -continuous if, and only if, the $\lambda(k)$'s are uniformly absolutely continuous with respect to the $\mu(k)$'s, we obtain the following corollary.

Corollary 6.2. *Suppose that $\mathbb{M}(k)$ is an increasing sequence of lattices of subsets of Ω . For each positive integer k , suppose each of $\lambda(k)$ and $\mu(k)$ is a nonnegative additive set function defined on $\mathcal{Q}(k)$, the algebra generated by \mathbb{M}_k , such that $\lambda(k+1)|_{\mathcal{Q}(k)} = \lambda(k)$, $\mu(k+1)|_{\mathcal{Q}(k)} = \mu(k)$, $\lambda(k) \in V^\Phi(\Omega, \mathcal{Q}(k), \mu(k), R)$, and $I_\Phi(\lambda(k)) \leq M$. If $\mathcal{Q} = \bigcup_{k=1}^\infty \mathcal{Q}(k)$, then $\lambda(k)$ and $\mu(k)$ are in $V^\Phi(\Omega, \mathcal{Q}, \mu, R)$, where $\mu(E) = \mu(k)(E)$ if E is in $\mathcal{Q}(k)$. Finally, suppose that the $\lambda(k)$'s are uniformly absolutely continuous with respect to the $\mu(k)$'s. Then the sequence $\{H_k\}$ converges in norm.*

Note. With $\Phi(x) = |x|$ (i.e. Darst's result) the requirement that $I_\Phi(\lambda(k)) \leq M$ is superfluous because in this case λ absolutely continuous with respect to μ

implies that λ is of bounded variation (cf. [10, Theorem 1, p. 529]).

VII. Vector valued set functions. In this section we use Definition 4.2 coordinatewise to define the Radon-Nikodym derivative of a bounded additive vector valued set function, $\lambda: \mathfrak{M} \rightarrow R^n$, with respect to a nonnegative bounded additive set function, $\mu: \mathfrak{M} \rightarrow R$, on a lattice \mathfrak{M} of subsets of a nonempty set Ω . Then the "best approximation" theorem, Theorem 5.5, and the martingale convergence theorem, Theorem 6.1, are extended to this case. We begin by introducing some notation.

If $x \in R^n$, then $x = \sum_{i=1}^n x_i e_i$, where $e_i = (e_{i1}, \dots, e_{in})$, with $e_{ij} = 0$ for $i \neq j$ and $e_{ii} = 1$ for $i = j$. We will use the norm (recall that all norms on R^n are equivalent)

$$\|x\| = \sum_{i=1}^n |x_i|.$$

We partially order R^n by the order relation " \leq ", where $x \leq y$ if $x_i \leq y_i$, $i = 1, \dots, n$. Then $|x| = \sup(x, -x) = \sum_{i=1}^n |x_i| e_i$ and $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$ (i.e., R^n with this ordering is a Banach lattice).

If $f: \Omega \rightarrow R^n$, then $f(x) = \sum_{i=1}^n f_i(x) e_i$, where $f_i: \Omega \rightarrow R$.

Definition 7.1. A function $f: \Omega \rightarrow R^n$ is said to be \mathfrak{M} -measurable if each coordinate function $f_i: \Omega \rightarrow R$, $i = 1, \dots, n$, is \mathfrak{M} -measurable (Definition 2.3); i.e., for all $a \in R^n$, $[f_i > a_i] \in \mathfrak{M}$, $i = 1, \dots, n$.

Let $P^n = \{x \in R^n: \theta \leq x, \text{ where } \theta = (0, \dots, 0)\}$ and suppose that $\lambda: \mathfrak{M} \rightarrow P^n$ and $\mu: \mathfrak{M} \rightarrow P^1$ are bounded additive (Definition 2.2) set functions. The unique extensions [11, Corollary 1.2.1, p. 189] of λ and μ to the algebra \mathfrak{Q} , generated by the lattice \mathfrak{M} , are also denoted by λ and μ . Then

$$\lambda(E) = \sum_{i=1}^n \lambda_i(E) e_i, \quad E \in \mathfrak{Q},$$

where $\lambda_i: \mathfrak{Q} \rightarrow P^1$ is a bounded additive function on \mathfrak{Q} , $i = 1, \dots, n$.

Definition 7.2. The Radon-Nikodym derivative of λ with respect to μ on \mathfrak{M} is a L_1 -Cauchy sequence $F = \{f_k\}_{k=1}^\infty$ of simple \mathfrak{M} -measurable vector valued functions, where $f_k(x) = \sum_{i=1}^n f_{i,k}(x) e_i$ and $F_i = \{f_{i,k}\}_{k=1}^\infty$ is the derivative of λ_i with respect to μ on \mathfrak{M} (Definition 4.2). Thus $F(E) = \sum_{i=1}^n F_i(E) e_i$, $E \in \mathfrak{Q}$.

The sequence $F = \{f_k\}_{k=1}^\infty$ is a L_1 -Cauchy sequence because

$$\int_{\Omega} \|f_N(x) - f_M(x)\| d\mu = \sum_{i=1}^n \int_{\Omega} |f_{i,N}(x) - f_{i,M}(x)| d\mu \rightarrow 0.$$

Thus $F \in V_1(\Omega, \mathfrak{Q}, \mu, R^n)$.

Suppose now that \mathfrak{U} is an algebra of subsets of Ω , that $\mathfrak{M} \subset \mathfrak{U}$, and that $\mu: \mathfrak{U} \rightarrow P^1$ is a bounded additive function. Recall that $H \in V^\Phi(\Omega, \mathfrak{U}, \mu, R^n)$ can be identified with a sequence $\{b_k\}$ of simple \mathfrak{U} -measurable vector valued functions

such that $\int_{\Omega} \Phi(\|b_k(x) - b_m(x)\|) d\mu \rightarrow 0$ (as $k, m \rightarrow \infty$).

Definition 7.3. If λ is the restriction to \mathfrak{M} of $H \in V^{\Phi}(\Omega, \mathfrak{U}, \mu, R^n)$, then we denote the derivative, $F = \{f_k\}_{k=1}^{\infty}$, of λ with respect to μ on \mathfrak{M} by $F = E(H, \mathfrak{M})$.

If $H \in V^{\Phi}(\Omega, \mathfrak{U}, \mu, R^n)$, then $H(E) = \sum_{i=1}^n H_i(E) e_i$, where $H_i \in V^{\Phi}(\Omega, \mathfrak{U}, \mu, R)$. Thus

$$F(A) = E(H, \mathfrak{M})(A) = \sum_{i=1}^n E(H_i, \mathfrak{M})(A) e_i = \sum_{i=1}^n F_i(A) e_i, \quad A \in \mathfrak{U}.$$

The sequence $F = \{f_k\}_{k=1}^{\infty}$ is a L^{Φ} -Cauchy sequence because the convexity of Φ and the Δ_2 -condition imply that

$$\int_{\Omega} \Phi(\|f_N(x) - f_M(x)\|) d\mu \leq C \sum_{i=1}^n \int_{\Omega} \Phi(|f_{i,N}(x) - f_{i,M}(x)|) d\mu \rightarrow 0.$$

Hence $F \in V^{\Phi}(\Omega, \mathfrak{U}, \mu, R^n)$, i.e., the derivative of a V^{Φ} -function is a V^{Φ} -function.

Theorem 7.4. The derivative of a function $H \in V^{\Phi}(\Omega, \mathfrak{U}, \mu, R^n)$ is a function $F = E(H, \mathfrak{M}) \in V^{\Phi}(\Omega, \mathfrak{U}, \mu, R^n)$.

That $F = E(H, \mathfrak{M})$ is the best V_2 -approximation to $H \in V_2(\Omega, \mathfrak{U}, \mu, R^n)$ by a L_2 -Cauchy sequence of simple \mathfrak{M} -measurable functions follows immediately from Theorem 5.5.

Theorem 7.5. If $H \in V_2(\Omega, \mathfrak{U}, \mu, R^n)$ and $F = E(H, \mathfrak{M})$, then

$$(7.5.1) \quad F \in V_2,$$

$$(7.5.2) \quad \int HK d\mu \leq \int FK d\mu, \quad K \text{ } \mathfrak{M}\text{-measurable}, K \in V_2,$$

$$(7.5.3) \quad \int HK d\mu \geq \int FK d\mu, \quad K \text{ } \mathfrak{M}^c\text{-measurable}, K \in V_2,$$

$$(7.5.4) \quad \int HF d\mu = \int F^2 d\mu,$$

$$(7.5.5) \quad \int (H - F)^2 d\mu \leq \int (H - K)^2 d\mu, \quad K \text{ } \mathfrak{M}\text{-measurable}, K \in V_2.$$

Moreover, the conditions of the theorem characterize F among the L_2 -Cauchy sequences.

Proof. We simply apply Theorem 5.5 to each coordinate. For example,

$$\int HK d\mu = \sum_{i=1}^n \int H_i K_i d\mu \leq \sum_{i=1}^n \int F_i K_i d\mu = \int FK d\mu.$$

Since $\|H\|_{\Phi} = \|\sum_{i=1}^n H_i e_i\|_{\Phi} \leq \sum_{i=1}^n \|H_i\|_{\Phi}$ for $H \in V^{\Phi}(\Omega, \mathfrak{U}, \mu, R^n)$, we can obtain a martingale convergence theorem by applying Theorem 5.1 coordinate-wise. To this end, we first establish the necessary notation.

Suppose that $\{\mathfrak{M}_k\}$ is an increasing sequence of lattices of subsets of Ω , and $\mathfrak{M} = \bigcup_k \mathfrak{M}_k$. Then the algebras \mathcal{Q}_k of subsets of Ω that are generated by these lattices increase to the algebra \mathcal{Q} generated by the lattice \mathfrak{M} .

Suppose that $\lambda: \mathcal{Q} \rightarrow P^n$ and $\mu: \mathcal{Q} \rightarrow P^1$ are bounded additive functions on \mathcal{Q} , and that $\lambda \in V^\Phi(\Omega, \mathcal{Q}, \mu, R^n)$. Denote by λ^k and μ_k the restrictions of λ and μ to \mathcal{Q}_k . Then λ^k is in $V^\Phi(\Omega, \mathcal{Q}_k, \mu_k, R^n)$. Let $E(\lambda, \mathfrak{M}) = F = \{f_m\}_{m=1}^\infty$ and $E(\lambda^k, \mathfrak{M}_k) = G^k = \{g_{k,m}\}_{m=1}^\infty$. By Theorem 7.4, F and G^k are in V^Φ .

Because the sequences $G^k = \{g_{k,m}\}_{m=1}^\infty$ are Cauchy sequences in $L^\Phi(\Omega, \mathcal{Q}_k, \mu_k, R^n)$ they are Cauchy sequences in $L^\Phi(\Omega, \mathcal{Q}, \mu, R^n)$ (i.e. $\int_\Omega \Phi(\|g_{k,l}(x) - g_{k,m}(x)\|) d\mu = \int_\Omega \Phi(\|g_{k,l}(x) - g_{k,m}(x)\|) d\mu_k$). Hence the equation

$$H^k(E) = \lim_{m \rightarrow \infty} \int_E g_{k,m} d\mu, \quad \text{for } E \in \mathcal{Q},$$

defines an additive extension of G^k to \mathcal{Q} . Notice that $\{H^k\}$ is determined by $\{\mathfrak{M}_k\}$, λ , and μ . We call $\{H^k, \mathfrak{M}_k\}$ a martingale.

Theorem 7.6. *The sequence $\{H^k\}$ converges to F in $V^\Phi(\Omega, \mathcal{Q}, \mu, R^n)$.*

Proof. Suppose $\lambda(E) = \sum_{i=1}^n \lambda_i(E) e_i$, then $\lambda^k(E) = \sum_{i=1}^n \lambda_i^k(E) e_i$, where λ_i^k is the restriction of λ_i to \mathcal{Q}_k . Set $F_i = E(\lambda_i, \mathfrak{M})$, $G_i^k = E(\lambda_i^k, \mathfrak{M}_k) = \{g_{i,k,m}\}_{m=1}^\infty$, and

$$H_i^k(E) = \lim_{m \rightarrow \infty} \int_E g_{i,k,m} d\mu, \quad E \in \mathcal{Q}.$$

Then by Theorem 6.1, for $i = 1, \dots, n$,

$$\|H_i^k - F_i\|_\Phi \rightarrow 0 \quad (\text{as } k \rightarrow \infty).$$

Finally, since $H^k(E) = \sum_{i=1}^n H_i^k(E) e_i$ and $F(E) = \sum_{i=1}^n F_i(E) e_i$,

$$\|H^k - F\|_\Phi \leq \sum_{i=1}^n \|H_i^k - F_i\|_\Phi \rightarrow 0 \quad (\text{as } k \rightarrow \infty).$$

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